

CONNECTION PROBABILITIES FOR CONFORMAL LOOP ENSEMBLES

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ABSTRACT. The goal of the present paper is to explain, based on properties of the conformal loop ensembles CLE_κ (both with simple and non-simple loops, i.e., for the whole range $\kappa \in (8/3, 8)$) how to derive the connection probabilities in conformal rectangles for a conditioned version of CLE_κ which can be interpreted as a CLE_κ with wired/free/wired/free boundary conditions on four boundary arcs (the wired parts being viewed as portions of to-be-completed loops). In particular, in the case of a conformal square, we prove that the probability that the two wired sides hook up so that they create one single loop is equal to $1/(1 - 2\cos(4\pi/\kappa))$.

Comparing this with the corresponding connection probabilities for discrete $O(N)$ models for instance indicates that if a dilute $O(N)$ model (respectively a critical $\text{FK}(q)$ -percolation model on the square lattice) has a non-trivial conformally invariant scaling limit, then necessarily this scaling limit is CLE_κ where κ is the value in $(8/3, 4]$ such that $-2\cos(4\pi/\kappa)$ is equal to N (resp. the value in $[4, 8)$ such that $-2\cos(4\pi/\kappa)$ is equal to \sqrt{q}).

Our arguments and computations build on the one hand on Dubédat's SLE commutation relations (as developed and used by Dubédat, Zhan or Bauer-Bernard-Kytölä) and on the other hand, on the construction and properties of the conformal loop ensembles and their relation to Brownian loop-soups, restriction measures, and the Gaussian free field, as recently derived in works with Sheffield and with Qian.

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1. INTRODUCTION

1.1. A motivation from discrete models. The simplest and first way to argue that if critical bond percolation on the square lattice possesses a conformally invariant scaling limit (which is a strong assumption – recall that proving this is still an open question), then the scaling limit of the interfaces has to be the Schramm-Loewner evolution (SLE_κ) with parameter $\kappa = 6$ was due to Schramm [44, 45] in 1999 and it can be sketched as follows. First, the usual argument involving the conformal Markov property shows that the scaling limit of interfaces would have to be an SLE_κ

path for some $\kappa > 0$, so that it remains to pin down the actual value of κ . One simple property that allows one to do so is that if one considers an SLE_κ in a square from the bottom left corner to the top left corner, then one can compute the probability that it hits the right-hand side of the square along the way. It turns out that $\kappa = 6$ is the only value for which this probability is equal to $1/2$. On the other hand, for discrete critical bond percolation on a (quasi)-square $[0, n+1] \times [0, n]$ portion of the square grid, duality shows that the probability that there exists a left-to-right open crossing (which means that the discrete percolation interface analog of the previous SLE_κ will hit the right-hand side of the square) is equal to $1/2$ for all $N \geq 1$. One can then conclude that SLE_6 is the only possible candidate for a conformally invariant scaling limit of the percolation interface.

Note that SLE_6 also possesses other properties (see for instance the locality property derived in [27]) that also imply that it is the only possible conformally invariant scaling limit for critical percolation without even referring to any discrete crossing probability, but the above argument is already short, direct and convincing. Recall that in order to prove that percolation is indeed conformally invariant in the scaling limit, and that discrete percolation interfaces do converge to SLE_6 , a lot of further work is required. See the work of Smirnov [52] for the case of the triangular lattice (and for instance [58] for a survey of the proof of the convergence of interfaces).

Two of the most natural and classical classes of discrete models that are supposed to give rise to SLE curves in the scaling limit are the $O(N)$ models (both the dense and the dilute versions) and the critical bond $\text{FK}(q)$ -percolation models (we will briefly recall the definition of these models in Appendix A). On planar lattices, exactly three of these models have been proved to indeed converge to an SLE-based scaling limit: The Ising model (which is the $O(1)$ model, where the interfaces converge to SLE_3 paths), the FK-percolation models for $q = 2$ (this is the FK model related to the Ising model, where the interfaces converge to $\text{SLE}_{16/3}$ paths) – see [7, 21] and the references therein, or [54] for a survey), and the $\text{FK}(q)$ -percolation model in the limit $q = 0^+$ (this is the uniform spanning tree model, where its boundary Peano curve converges to SLE_8) – see [29].

It has been conjectured (see for instance [20, 43, 16]), based on the identification between exponents, probabilities or dimensions that one can rigorously compute for SLE processes on the one hand and the corresponding quantities that had been previously predicted from the methods of theoretical physics (conformal field theory, quantum gravity or Coulomb gas methods, see for instance [40, 6]) for the asymptotic behavior of the discrete models on the other hand, that the $O(N)$ models have a non-trivial and conformally invariant scaling limit for all $N \in (0, 2]$ and that this scaling limit should be related to SLE_κ curves for $N = -2 \cos(4\pi/\kappa)$, where $\kappa \in (8/3, 4]$ if one considers the dilute $O(N)$ model and $\kappa \in (4, 8)$ if one considers the dense $O(N)$ model. Similarly, the scaling limit of the critical $\text{FK}(q)$ -percolation model interfaces should be non-trivial for $q \in (0, 4]$ and described by SLE_κ curves, for $\sqrt{q} = -2 \cos(4\pi/\kappa)$, where $\kappa \in [4, 8)$ (mind of course that $\cos(4\pi/\kappa)$ is negative for all $\kappa \in (8/3, 8)$).

1.2. Content of the present paper. Recall that a CLE_κ (conformal loop ensemble) for $\kappa \in (8/3, 8)$ is a particular random collection of loops in a simply connected planar domain such that the loops in a CLE_κ are SLE_κ type loops. While the SLE_κ curves can be argued (via the conformal Markov property) to be the only possible conformally invariant scaling limit of single interfaces for a wide family of discrete interfaces for lattice models with well-chosen boundary conditions (that involve choosing two special points on the boundary of the domain, and choosing one boundary arc to be “wired” while the other one is “free”), the CLE_κ can be argued to be the only possible conformally invariant scaling limit for the joint law of *all* of the macroscopic interfaces for the same models with “uniform” free boundary conditions. For the aforementioned lattice models ($\text{FK}(q)$ -percolation and $O(N)$), this has now been proved for the same cases as for individual interfaces, see [29, 5, 23, 4])

The results of the present paper on CLE_κ connection probabilities provide a generalization of Schramm's original argument for percolation that we outlined above to all of these models. More specifically, for all $\kappa \in (8/3, 8)$, we will first explain how to define the law of a CLE_κ in a conformal rectangle, with “wired” boundary conditions on two opposite sides of the rectangle. The idea, is to start with the usual CLE_κ and to partially discover it starting from two boundary points; in other words, our CLE_κ with free/wired/free/wired boundary conditions will be defined as a conditioned version of the usual CLE_κ (even if one conditions on an event of zero probability). The wired portions of the boundary should be thought of as parts of to-be completed loops in this conditioned CLE_κ .

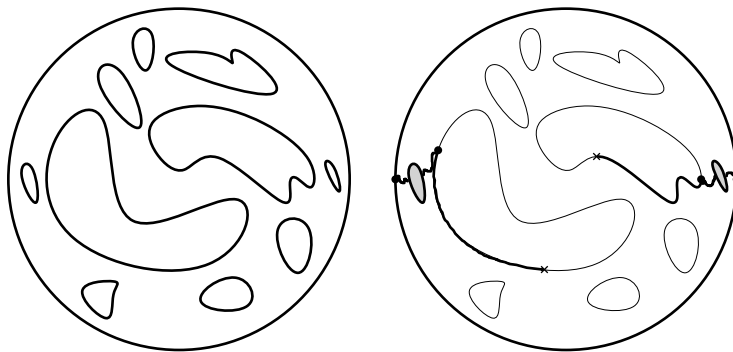


FIGURE 1. Discovering a simple CLE starting from two boundary points and creating a CLE with two wired boundary arcs (sketch). In this example, the two wired boundary arcs are not part of the same loops.

Then, for these CLE_κ with two wired boundary arcs, we will derive the probability that the two wired pieces are part of the same loop, as a function of κ and of the cross-ratio x of the four corners of the conformal rectangle.

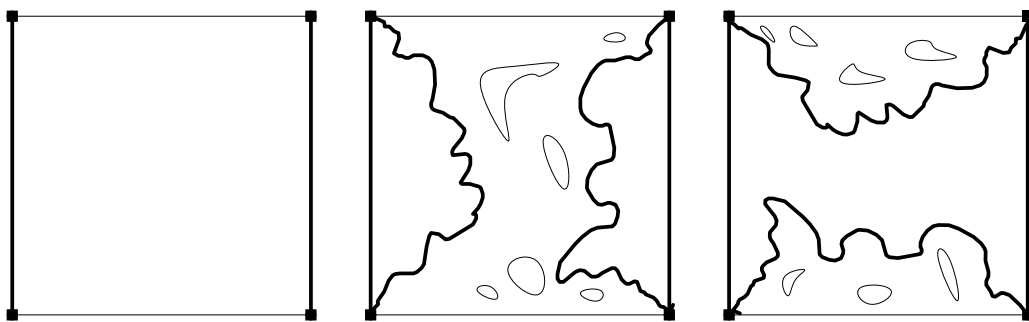


FIGURE 2. A square with wired vertical boundaries, and a sketch of the two possible connection configurations, together with the remaining outermost CLE loops

In particular, we will see that:

Theorem 1. *Consider $\kappa \in (8/3, 8)$ and a CLE_κ with two wired boundary arcs in a square, where the boundary arcs are two opposite sides of the square. Then, the probability that the boundary arcs hook up into one single loop is equal to $1/(1 - 2 \cos(4\pi/\kappa))$.*

Note that this connection probability (we will also sometimes use equivalently the term hook-up probability) in squares first decreases from 1 to $1/3$ when κ increases from $8/3$ to 4 (this is the regime

where the CLE_κ consists of simple disjoint loops), and then increases again from $1/3$ to 1 , when κ increases from 4 to 8 (which is the regime of non-simple loops). Hence, the hook-up probability in a square with the alternating boundary conditions is always at least $1/3$.

Deriving Theorem 1 is the core of our paper and the proofs will make no reference to discrete models. It has nevertheless some implications about the conjectural relation between conformal loop ensembles and discrete models that we now briefly discuss: One can compare it with the fact that for discrete $O(N)$ models (on any lattice with some symmetries) and for the critical FK(q) percolation model on \mathbf{Z}^2 , the corresponding discrete connection probabilities in squares (or some other given symmetric shape if one considers a lattice other than \mathbf{Z}^2) are equal to $1/(1+N)$ and $1/(1+\sqrt{q})$, independently of the size of the square, just because of a symmetry/duality argument (we will recall this in Appendix A). This can be viewed as the natural generalization to those models of the crossing probability of squares feature of critical percolation (which is the special case $q = 1$, and for percolation, boundary conditions do not matter). Hence, one gets the following conditional results, that generalize Schramm's 1999 statement for critical percolation and SLE_6 (note also that Theorem 7 in [15] shows rigorously that if an FK(q)-model scaling limit for $q < 4$ has a conformally invariant scaling limit, it would be boundary touching i.e., the value of κ would have to be greater than 4):

- If a dilute $O(N)$ model has a non-trivial and conformally invariant scaling limit consisting of simple loops, then necessarily $N \in (0, 2]$ and the scaling limit has to be CLE_κ for the value of $\kappa \in (8/3, 4]$ such that $N = -2 \cos(4\pi/\kappa)$.
- If a dense $O(N)$ model has a non-trivial and conformally invariant scaling limit consisting of non-simple loops, then necessarily $N \in (0, 2]$ and the scaling limit has to be CLE_κ for the value of $\kappa \in [4, 8)$ such that $N = -2 \cos(4\pi/\kappa)$.
- If a critical FK(q)-model on \mathbf{Z}^2 has a non-trivial conformally invariant scaling limit, then necessarily $q \in (0, 4]$ and the scaling limit has to be CLE_κ for the value of $\kappa \in [4, 8)$ such that $\sqrt{q} = -2 \cos(4\pi/\kappa)$.

1.3. Brief discussion of the related literature. Let us briefly discuss the relationship between the present contribution and some of the closely related results in the literature:

- Connection probabilities and related questions for families of SLE paths have been the focus of interesting mathematical work by Dubédat, Zhan, Bauer-Bernard-Kytölä and others (see for instance [9, 10, 11, 3, 63, 64, 65, 24, 25] and the references therein) that is very much relevant and related to the present paper, and that we will in fact use. Let us now very briefly explain how our contribution fits with respect to these references. In these papers, the main focus is on the description and classification of the joint law of “commuting” SLE curves, that start from a number of boundary points of a simply connected domain (these are Dubédat's commutation relations). In the particular case where one looks at four boundary points a_1, \dots, a_4 and assumes that all of the SLE curves are locally absolutely continuous with respect to SLE_κ curves for the same value of κ , these commutation relations allow one to describe all of the possibilities up until the curves hit. If one applies this to our precise setup, these results state that one has (for each value of κ) exactly a one-parameter family of possible candidates for the joint law of the four strands of our CLE_κ with two wired boundary conditions. Basically, one has two extremal solutions such that for the first one, the strand started from a_1 always ends up at a_2 , while for the second one, it ends up at a_4 (in both these cases, the law of the pair of strands are also known under several names: intermediate SLE processes, hypergeometric SLEs, or bichordal SLE processes) and in our setting, they describe the law of the strands, when one conditions on one (or the other)

connection events. In particular, if one *assumes* the value of the connection probability in a square, then these commutation ideas provide the exact form of the connection probability in terms of the cross-ratio of the conformal rectangle (and because this probability defines a martingales when one explores one strand, it also defines the exact description of the driving function of one strand), we will come back to this in Section 4. The purpose of the present paper is to provide an actual computation of this connection probability in conformal squares for our wired/free/wired/free boundary conditions, uniquely via CLE considerations and no conjectural relation to discrete models, which (we believe) is a new input.

- The hook-up probability in squares can be derived by other means for some special values of κ : As we have already mentioned, the fact that it is $1/2$ for CLE_6 is a direct consequence of the special target-invariance properties of SLE_6 . The fact that the hook-up probability is also $1/2$ when $\kappa = 3$ can be viewed as a consequence of the fact that CLE_3 is the scaling limit the Ising model [4] because of the symmetries in the Ising model (this observation already appears in [9, 3] – at that time the case $\kappa = 3$ was a conditional result since it had not yet been established that the Ising model converges to SLE_3 which is now established, see also the recent paper [62]). Similarly, the fact that the hook-up probability is $1/(1 + \sqrt{2})$ in the special case where $\kappa = 16/3$ can in fact be viewed as a consequence of the fact [22, 23] that $\text{CLE}_{16/3}$ is proved to be the scaling limit of the FK(2) model (see Appendix A for why the crossing probability for this discrete model is $1/(1 + \sqrt{2})$). Also, the fact that the hook-up probability tends to 1 as $\kappa \rightarrow 0^+$ and $\kappa \rightarrow 8^-$ could be inferred directly from the Brownian loop-soup description from [51] for the $\kappa \rightarrow 0^+$ case, and from the relation to uniform spanning trees when $\kappa \rightarrow 8^-$ (see [29]). As we will explain in Section 3, the fact that the connection probability is $1/3$ in the case where $\kappa = 4$ can be worked out using the relation between CLE_4 and the Gaussian Free Field.
- The CLE percolation approach that we developed with Sheffield [36] gives a continuous version of the relation between FK-percolation and the corresponding Potts models in terms of conformal loop ensembles CLE_κ and $\text{CLE}_{16/\kappa}$. In [38], combining the results of [36] with ideas from Liouville quantum gravity (LQG), we provide another way to get the relation $\sqrt{q} = -2\cos(4\pi/\kappa)$ out of CLE_κ considerations only (this time without any reference to discrete crossing probabilities). Note that this other CLE/LQG approach does not directly yield the value of the connection probabilities or the relation to the dilute $O(N)$ models, and that it is somewhat more elaborate than the present one. It is also related to the considerable progress that has been done in recent years in relating these questions to structures in random geometries/random maps (that can be viewed as trying to put some of the theoretical physics considerations onto firm mathematical ground), see for example [18, 50, 17] and the references therein.

Let us now make some further bibliographic comments on the conjectural relation with discrete models:

- It is interesting to see that the $q = 4$ and $N = 2$ thresholds for the nature of the phase transition of FK(q) percolation and $O(N)$ models show up from this CLE_κ end, via the fact that the lowest possible CLE_κ hook-up probability in conformal squares is $1/3$. Note that $q = 4$ has been recently proved (rigorously and based on the study of discrete models) to be the threshold for existence of a continuous phase transition for FK(q)-percolation models on \mathbb{Z}^2 [15, 14]. In particular, [14] shows rigorously that the scaling limit of the critical FK(q) model's interfaces are trivial when $q > 4$ (which is of course a much stronger statement than the conditional “then necessarily $q \in (0, 4]$ ” in the FK-part of our statement above).

- As we have already mentioned, these relations between q and κ , and between N and κ have appeared in numerous papers before. But, to the best of our knowledge, except for the specific particular cases of κ that we have already mentioned, they were not based on rigorous SLE-type considerations. More precisely, the argument was the following: If these models have a conformally invariant scaling limit, then it must be described by an SLE_κ for some κ . In order to identify which value of κ is the right one, they matched some computation of probabilities of events for SLE (or of critical exponents, or of dimensions) with the corresponding values that were *predicted* to be the correct ones for the scaling limit of the discrete model, based on theoretical physics considerations (see for instance [43] for the FK(q) conjecture based on the physics dimension predictions; another approach is related to the discrete parafermionic observable for FK models – see [53] or [13] for a detailed discussion and more references –, where the spin is defined as $\sigma = 1 - (2/\pi) \arccos(\sqrt{q}/2)$, and that is conjectured to correspond in the scaling limit to some SLE martingale, see for instance [61]). In the present paper, we identify the candidate value of κ using a feature that is rigorously known to hold in the discrete model (and therefore in its scaling limit, if it exists). So, in a way, one could view it as an SLE/CLE derivation of the conjectural relation between the lattice models and the corresponding conformal field theory (i.e., a relation between the q or N and the central charge $c = (6 - \kappa)(3\kappa - 8)/(2\kappa)$ which can be derived via the SLE restriction property [28] or the loop-soup construction of CLE [51]).
- In relation with the CLE percolation item mentioned above, let us just stress that the relation $(2/\kappa) + (2/\tilde{\kappa}) = 1$ between the values $\kappa \in (8/3, 4)$ and $\tilde{\kappa} \in (4, 8)$ that have the same connection probability is not at all the same as the $\kappa\kappa' = 16$ duality relation between SLE_κ and $\text{SLE}_{\kappa'}$ from [63, 10] or the Edwards-Sokal coupling between CLE_κ and $\text{CLE}_{\kappa'}$ derived in [36]. It is however possible to combine the present work or the results of [38] with the CLE percolation results of [36] – this for instance indicates based on CLE considerations only that if the scaling limit of the critical FK-percolation model for $q = 3$ on \mathbf{Z}^2 is non-trivial and conformally invariant, then the scaling limit of the critical Potts model for $q = 3$ would exist as well and could be described in terms of CLE_κ for the value of $\kappa \in (8/3, 4]$ such that $\sqrt{q} = -2 \cos(4\pi\kappa/16) = -2 \cos(\pi\kappa/4)$ i.e., $\kappa = 10/3$.

1.4. Structure of the paper. Let us describe the structure of the paper, and explain where we use which results from other papers:

In Section 2, we will first recall some background material about CLEs and their properties, and then define what we will call CLE with two wired boundary arcs. As we will explain, this builds on the shoulders of some previous work: The definition and conformal invariance of CLEs for $\kappa \in (8/3, 8)$ from [49, 51, 34], the exploration features of CLE as studied in [59, 36], and last but not least, the conformal invariance of hook-up probabilities as derived in [37]. This is a section where we use directly and indirectly various background material from the family of papers that construct conformal loop ensembles and their basic properties. Those readers who want to focus on the computational part of the derivation of Theorem 1 can choose to take the results of that section for granted.

In Section 3, which can be viewed as a brief interlude, we explain how it is possible to prove Theorem 1 in the case where $\kappa = 4$ using the coupling between the Gaussian free field and CLE_4 (from [31], see also [2]).

In Section 4, we briefly survey what the aforementioned works on commutation relations [9, 10, 11, 3, 63, 64, 65] do imply in our set-up of wired CLEs.

In Section 5, we describe the main steps of our proofs of Theorem 1, separately in the cases $\kappa \in (8/3, 4]$ and $\kappa \in (4, 8)$, that allow us to reduce the determination of the connection probability to actual concrete estimates of probabilities of events that we then derive in Sections 6 and 7. In the case of simple CLEs, the arguments in Section 5 will rely on the loop-soup construction of CLE_κ , and more specifically on the decomposition of loop-soup clusters and the relation to restriction measures, as studied in [42, 41]. The actual computations in Sections 6 and 7 will involve some considerations involving SLE and hypergeometric functions.

We conclude with two short appendices, recalling very briefly some basics about $O(N)$ models and their connection probabilities, and about the properties of hypergeometric functions that we are using in our proofs.

2. DEFINING CLE WITH TWO WIRED BOUNDARY ARCS

2.1. CLE Background and explorations. Let us quickly recall some features of conformal loop ensembles (CLE) – the reader may wish to consult [36] for background and further references. For our purpose, it will be sufficient to focus on their non-nested versions. CLE was defined in [49] as the natural candidate that should describe the joint law of outermost interfaces in a number of critical models from statistical physics in their scaling limit. Sheffield’s construction is based on the target-invariance (see [9, 48]) of variants of SLE_κ , the $\text{SLE}_\kappa(\kappa - 6)$ processes that can be defined nicely for all $\kappa \in (8/3, 8)$ (when $\kappa \in (8/3, 4]$, one has to consider $\text{SLE}_\kappa(\kappa - 6)$ processes with either Lévy compensation or side-swapping, because $\kappa - 6 < -2$ which corresponds to the dimension of a certain Bessel process being strictly smaller than 1; see [49, 36, 59] for background). This property enables one to define for each simply connected domain D and each boundary point x_0 , a branching tree of such processes, starting from x_0 and that target all points in D . Branches of the tree trace loops along the way, so that this branching tree defines a random collection of loops, that Sheffield called CLE_κ . Note that with this construction, the law of this CLE_κ seems to depend on the choice of the boundary point.

In order to prove that this law does not depend on the choice of x_0 and therefore prove that the CLEs constructed by Sheffield are indeed a random collection of loops whose laws are invariant under the whole group of conformal automorphisms of D , some non-trivial arguments are needed:

- When $\kappa \in (8/3, 4]$, which is the case where the CLE loops are all disjoint and do not touch the boundary of D , the proof in [51] uses the Brownian loop-soup in D introduced in [30], which is a natural Poisson point process of Brownian loops in D , with intensity given by a constant c times a certain natural measure on Brownian loops. Note that the loops in a loop-soup can be thought of as being independent, so that they can overlap and therefore create clusters of Brownian loops. The fact that the outer boundaries of loop-soup clusters would give rise to random collections of SLE-type loops that behave nicely under perturbations of the domains that they are defined in had been outlined in 2003 [55].

As it turns out, it has then been proved in [51] that in fact: (a) for all $c \leq 1$, if one considers the outermost boundaries of these Brownian loop-soup clusters, one obtains a countable conformally invariant collection of mutually disjoint simple loops and (b) that for all x_0 , this collection of loops coincides with the CLE_κ defined by the branching tree construction. Since the loop-soup construction of CLE does not involve the boundary point x_0 , this therefore proves that the law of the CLE_κ defined by the branching tree construction is indeed independent of x_0 . The fact that one has these two different descriptions of the same CLE_κ (via the Brownian loop-soup or via branching $\text{SLE}_\kappa(\kappa - 6)$ processes) is a very useful fact when one tries to derive further properties of the conformal loop ensembles, and in the present paper, we will in fact use both these constructions in our proofs.

- When $\kappa \in (4, 8)$, some aspects of the branching $\text{SLE}_\kappa(\kappa - 6)$ tree are in some way simpler than for $\kappa \leq 4$. This makes it possible to view the conformal invariance of the CLE_κ as a direct consequence of the reversibility of SLE_κ processes. This reversibility for non-simple SLE paths is a non-trivial fact, that has been established in [34] using the connection with the Gaussian free field (GFF) and more precisely via the “imaginary geometry” approach developed in [32, 33, 34, 35]. We refer to [36, 37] for more details about the construction of CLE_κ for $\kappa \in (4, 8)$.

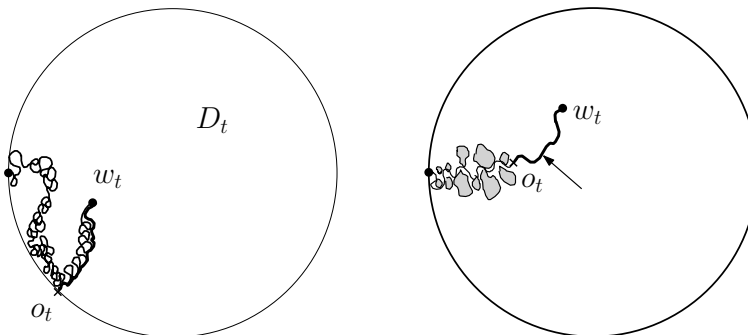


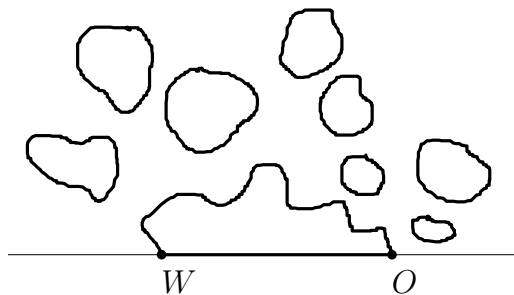
FIGURE 3. Sketch of discovery processes (for non-simple and simple CLEs) in the unit disk stopped while tracing a loop. The bold arc denotes the “wired” portion of D_t . The arrow indicates the “inside” part of the partially traced simple loop which corresponds to this wired part.

The branching tree description defines immediately what one can refer to as CLE_κ with one wired boundary arc. Suppose that one grows an $\text{SLE}_\kappa(\kappa - 6)$ process γ targeting some other boundary point y (for convenience, let us take the usual normalization where $x_0 = 0$, $y = \infty$ and $D = \mathbf{H}$), and that one stops this branch of the branching SLE tree at a (deterministic or stopping) time t , while it is in the middle of tracing a loop. We can then ask what is the conditional distribution of the CLE_κ in the unbounded connected component H_t of $\mathbf{H} \setminus \gamma[0, t]$. If we map back H_t onto the upper half-plane via the usual conformal transformation g_t normalized at infinity by $g_t(z) = z + o(1)$, then the image $g_t(\gamma_t)$ of the tip of the curve is the driving point of the Loewner chain W_t , while the other end of the CLE loop that one is currently tracing is mapped onto the marked point O_t that one uses in order to define the $\text{SLE}_\kappa(\kappa - 6)$ process. The target-invariance and the properties of the conformal loop-ensembles immediately show that the image under g_t of this conditional distribution can be described as follows (this works for all $\kappa \in (8/3, 8)$):

- First finish the currently traced loop by sampling an ordinary SLE_κ in \mathbf{H} from W_t to O_t .
- Then sample independent CLEs in the connected components of the complement of the SLE that are “outside” of the loop obtained by concatenating the SLE with the segment from O_t to W_t .

This description is valid for all $\kappa \in (8/3, 8)$ and it is clearly a conformally invariant function of (\mathbf{H}, W_t, O_t) . This distribution is what we will call CLE with one wired boundary arc (here the wired boundary arc of \mathbf{H} is the compact segment between W_t and O_t). By conformal invariance, this can then also be defined for any simply connected domain D and boundary arc ∂ . Examples are sketched in Figure 4 in the case $\kappa \in (8/3, 4]$.

As explained for instance in [51] or [59], one can also use procedures other than the exploration tree to discover the loops of a CLE_κ in a “Markovian” way. This includes for instance deterministic explorations such as discovering one after the other an in their order of appearance, all the CLE

FIGURE 4. Sketch of (part of) a simple CLE with wired arc $[W, O]$.

loops that intersect a given deterministic curve that starts on the boundary; for example in the unit disk, start from 1, and trace one after the other all loops that intersect the segment $[-1, 1]$ starting from 1, until one hits the imaginary axis. This last procedure will sometimes jump along the imaginary axis, but the CLE property will ensure that the previously defined wired CLE describes also the conditional law of the CLE when one stops the exploration in the middle of a loop. Such an exploration can be useful for $\kappa \in (8/3, 4]$ because it is a deterministic function of the CLE (which is actually not the case when $\kappa < 4$ for the branching tree, see [37]).

Another important tool for us will be the recent results about the decomposition of Brownian loop-soup clusters [42, 41]. In particular, we will use the following fact (see [41]): Consider a CLE_κ \mathcal{C} for $\kappa \in (8/3, 4]$ in the unit disk, that was obtained from a Brownian loop-soup \mathcal{L} with intensity c (as explained in [51]), and one starts an exploration from the boundary point -1 and stops it at some time t , when one is in the middle of tracing a loop. Then, we have seen that in the remaining to be discovered domain D_t , the conditional distribution of the CLE is a wired CLE, which is wired on the currently traced arc ∂_t . It turns out that it is in fact possible to describe the following aspects of the conditional distribution of the loop-soup itself in D_t .

- The outer boundary of the union of all the Brownian loops that do intersect ∂_t form a restriction measure with exponent $\alpha = (6 - \kappa)/(2\kappa)$ attached to this arc (see [28, 56] for background and definitions about restriction measures).
- The set of Brownian loops in D_t that do not intersect ∂_t form an independent loop-soup in D_t .

A feature that is worth mentioning already and that follows from this last item, is that if we stop the exploration at a stopping time such that the exploration has not exited the η -neighborhood of -1 , then this exploration will be independent of the set of Brownian loops in the loop-soup that do not intersect this neighborhood.

2.2. CLE with two “wired” boundary arcs. In the case where $\kappa \in (4, 8)$, we recall that the $\text{SLE}_\kappa(\kappa - 6)$ process is a deterministic function of the CLE_κ that it constructs. It follows local deterministic rules in order to discover (portions of) the loops that intersect the boundary of the domain. As explained in [37], this makes it possible for a given realization of the CLE in a simply connected domain, to choose two distinct boundary points, and to start discovering the CLE from both points, and to stop the two explorations at some fixed or random stopping time. Suppose that we follow this procedure and to choose to explore anticlockwise along the boundary (tracing the loops clockwise) for the first exploration, and to explore clockwise along the boundary (tracing the loops anticlockwise) for the second exploration, and that we stop them at some times during which they are both tracing loops and that they have not yet hooked up. Denote the remaining to be explored domain by D_t and the four marked boundary points (corresponding to the special points



FIGURE 5. An exploration of the CLE with the encountered Brownian loops (left). Adding the remaining Brownian loops completes the CLE (right).

of each of the explorations) (w_t, o_t, o'_t, w'_t) with obvious notations (see Figure 6). Then, as explained in Lemma 3.1 of [37], the conditional distribution of the CLE in D_t is then conformally invariant with respect to the configuration $(D_t, w_t, o_t, o'_t, w'_t)$. Let us insist on the fact that this conformal invariance statement is not trivial to prove, even if it seems intuitively obvious (one should keep in mind that the definitions of CLE themselves are not straightforward at all). This is what we will refer to as the CLE_κ in D_t with the wired arcs $w_t o_t$ and $o'_t w'_t$.

When $\kappa \in (8/3, 4]$, the exploration tree that discovers CLE_κ loops is not a deterministic function of the CLE (it involves necessarily side-swapping when $\kappa = 4$, and when $\kappa < 4$, there is also a lot of randomness lying in the trunk of the $\text{SLE}_\kappa(\kappa - 6)$ exploration tree – see [36] for all these facts). However, an important feature is that the trunk of an $\text{SLE}_\kappa(\kappa - 6)$ traces what we called a “conformal percolation interface” in the complement of the CLE that is defined locally, which enables for a given CLE, to launch two such explorations from two different boundary points, that are conditionally independent given the CLE.

To be specific, let us choose two given such $\text{SLE}_\kappa(\kappa - 6)$ explorations starting from two given boundary points, that are conditionally independent given the CLE. This requires for each of the two explorations to choose its side-swapping parameter $\beta \in [-1, 1]$, resp. β' (or their drift parameters μ and μ' when $\kappa = 4$). We impose the constraint that $\beta = -\beta'$ (or $\mu = -\mu'$ when $\kappa = 4$). Note that this in particular implies that it happens with positive probability that η and η' are exploring loops in different orientations (clockwise versus counterclockwise). Let us now stop these two explorations at some times during which they are both tracing loops and before they have actually hooked up. Denote the remaining to be explored domain with the four marked boundary points by D_t and (w_t, o_t, o'_t, w'_t) as before (see Figure 6).

Then, as explained in Lemma 3.6 of [37], on the event where η and η' are currently tracing loops in opposite directions (note that this is for instance almost surely the case if one chooses $\beta = 1$ and $\beta' = -1$), the conditional distribution of the CLE in D_t is then conformally invariant with respect to the configuration $(D_t, w_t, o_t, o'_t, w'_t)$. Again, this conformal invariance statement is not trivial to prove, even if it seems intuitively obvious. One slightly subtle point that is worth stressing is that the proof in [37] shows this conformal invariance statement for each chosen value of β (resp. μ) separately (in other words, it does not show that the law on CLE with two wired arcs does not depend on the chosen β), so that we will now work with one specific choice of exploration procedure, for instance for the symmetric side-swapping $\text{SLE}_\kappa(\kappa - 6)$ (i.e., $\beta = \beta' = 0$ and $\mu = \mu' = 0$ if $\kappa = 4$).

One can note that the results of the present paper will in fact provide as a by-product a proof of the fact that this law of CLE with two wired arcs does in fact not depend on the actual choice of β or μ .

To sum up things: For each $\kappa \in (8/3, 8)$ and for each rectangle $[0, L] \times [0, 1]$, there exists a law P_L on configurations in this rectangle, such that for any given previous discovery, the conditional law of the CLE in D_t is the conformal image of P_L where L is the value such that D_t and the four boundary points get mapped to the four corners of the rectangle. We will call these distributions P_L the CLE with wired boundary conditions on the two vertical boundaries of the rectangle, and $P_{D, a_1, a_2, a_3, a_4}$ will denote the law of the configuration in a simply connected domain with four distinct marked boundary points a_1, \dots, a_4 .

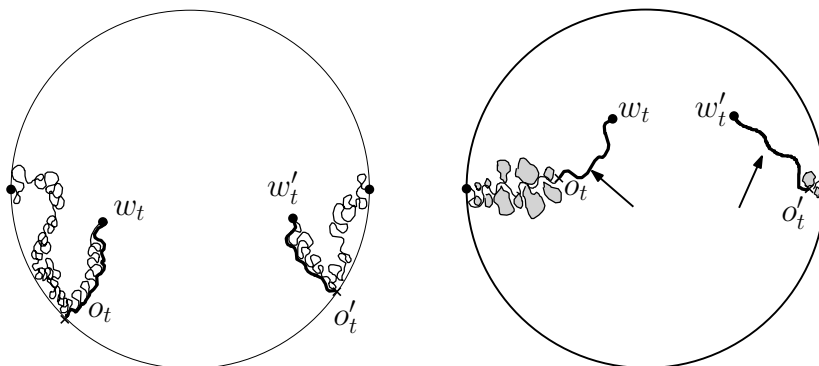


FIGURE 6. Exploring CLEs from both sides and creating the CLEs with two wired boundaries.

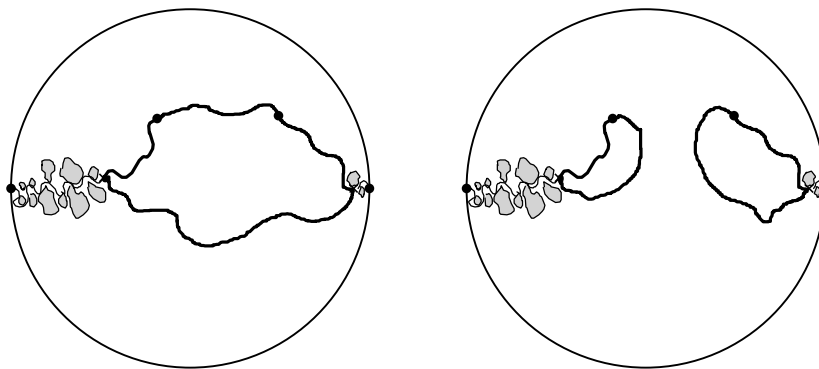


FIGURE 7. The two connection possibilities for the partially explored simple CLE from Figure 6.

The law P_L can be described in two steps:

- One can first complete the strands that start from the four corners. This will complete the loop(s) (which turns out to be one loop or two loops, depending on how the strands hook-up) that one had partially discovered, see Figure 7. Note that we have however not (yet) described at this point how to sample them.
- Then, in the remaining domain (outside of the traced loops), one samples independent CLEs.

This shows that in order to fully describe P_L , it is in fact sufficient to describe the law of the strands. Let us already mention that we will recall in the next section that Dubédat's commutation relation arguments (or bichordal SLE arguments) will do this to a certain extent. They basically

show that once one knows the hook-up probabilities (i.e., the probability that the four strands hook-up in the way to create one loop, as a function of L , see Figure 7 for the two possible options), then one can deduce the joint law of the strands. The main purpose of this paper will be precisely to determine this hook-up probability.

3. THE SPECIAL CASE $\kappa = 4$ AND THE GFF

Before studying the general case, let us briefly explain how Theorem 1 can be derived directly and easily when $\kappa = 4$, using the relation between CLE_4 and the GFF.

Recall that SLE_4 can be viewed as a level line of the Gaussian Free Field [46, 12]. The corresponding coupling of CLE_4 with the GFF was introduced in [31] (see also [2] for details) and can be described as follows: Sample one CLE_4 in a simply connected domain D , and toss an independent fair Bernoulli coin $\epsilon_j \in \{-1, 1\}$ for each CLE loop γ_j . Then, in the domains encircled by each of these loops, sample an independent GFF Γ_j with zero boundary conditions on γ_j (the GFF Γ_j is equal to 0 in the outside of γ_j). Then, for a certain choice of $\lambda > 0$, the field

$$\Gamma := \sum_j (2\epsilon_j \lambda + \Gamma_j)$$

is exactly a GFF in D . Furthermore, the CLE_4 and the labels (ϵ_j) are deterministic functions of the obtained field, and the side-swapping exploration of the CLE can be viewed as a deterministic function of the CLE and of the labels (ϵ_j) (see [36, 2] for details).

Suppose now that we explore the CLE_4 starting from two distinct boundary points in such a way that the two side-swapping $\text{SLE}_4(-2)$ explorations are conditionally independent given the CLE_4 (this can for instance be done by using two independent i.i.d. collections (ϵ'_j) and (ϵ''_j) and to view the $\text{SLE}_4(-2)$ processes as deterministic functions of the two corresponding GFFs, see [36] and the references therein), and stop these two explorations along the way as described above. The conditional distribution of the CLE_4 in the remaining to be discovered domain is then given by the CLE with two wired boundary arcs.

We also assume that we have chosen to stop our explorations in such a way that the remaining to be explored domain with the four marked points is exactly a conformal square (we can do for instance do this by first stopping the first exploration at some deterministic time, and then to stop the second one at the first time at which the configuration is a conformal square). Let us denote by E_1 the event that the four strands are hooked up so that they create a single CLE_4 loop (of the original CLE_4), and by E_2 the event that the four boundary strands are hooked up in the way that will create two disjoint CLE_4 loops.

Let us couple the CLE_4 with a GFF Γ as above, in such a way that the collection (ϵ_j) used to define the GFF are conditionally independent of the collections (ϵ'_j) and (ϵ''_j) that we used to define the explorations. On top of the partial discovery of the CLE_4 , we can also discover the corresponding boundary values of Γ . In other words, we can also decide to see whether on the two wired arcs, the GFF boundary values are $+2\lambda$ or -2λ . Let \tilde{E} denote the event that these boundary values are the same on both arcs.

Now we can note that $\mathbf{P}[\tilde{E}|E_1] = 1$ and $\mathbf{P}[\tilde{E}|E_2] = 1/2$ because of the rules that determine the GFF given the CLE (i.e., the coin flips are i.i.d. fair Bernoulli). On the other hand, it is known [31, 2] that the CLE_4 is a deterministic function of the GFF, so that conditioning on the joint information of the CLE and the GFF is the same as conditioning on the GFF only.

But, on the event \tilde{E} where the two boundary values on the partially explored strand are equal, we are looking at a GFF in a conformal square with boundary conditions $2\lambda, 0, 2\lambda, 0$ or $-2\lambda, 0, -2\lambda, 0$

on the four arcs. Hence, by symmetry,

$$\mathbf{P}[E_1|\tilde{E}] = \mathbf{P}[E_2|\tilde{E}] = 1/2.$$

This implies that

$$\mathbf{P}[E_1] = \mathbf{P}[\tilde{E} \cap E_1] = \mathbf{P}[\tilde{E}]\mathbf{P}[E_1|\tilde{E}] = \mathbf{P}[\tilde{E}]\mathbf{P}[E_2|\tilde{E}] = \mathbf{P}[\tilde{E} \cap E_2] = \frac{\mathbf{P}[E_2]}{2} = \frac{1 - \mathbf{P}[E_1]}{2}$$

and that the hook-up probability $\mathbf{P}[E_1]$ in the conformal square is indeed $1/3$ (i.e., $\theta = 2$).

Note that in this special case, we see that the marginal distributions of the four strands are in fact ordinary $\text{SLE}_4(\rho_1, \rho_2)$ processes (as opposed to the cases where $\kappa \neq 4$, where they turn out to be intermediate SLEs). This is also related to the fact that the hook-up probabilities that we will discuss in the next section take a very simple form in that case (which was already observed in the aforementioned papers by Dubédat or Bauer-Bernard-Kytölä).

4. CONSEQUENCES OF DUBÉDAT'S COMMUTATION RELATIONS

Let us consider again the general case $\kappa \in (8/3, 8)$, and let us now review what the results on commutation relations from [9, 10, 11, 63, 64, 65, 3] imply for our CLEs with two wired boundary arcs and for the hook-up probability as a function of the cross-ratio of the considered conformal rectangle. Let us define $H(x) = H_\kappa(x)$ for $x \in (0, 1)$ to be the probability that for a CLE_κ in the upper half-plane \mathbf{H} with two wired boundary arcs on $(-\infty, 0)$ and $(1 - x, 1)$, the two wired arcs are joined in such a way that they form one single loop (in other words, the strand starting from 0 ends at $1 - x$).

In the CLE_κ setup that we consider, for each choice of a simply connected domain D with four distinct points $a_1, \dots, a_4 \in \partial D$ ordered counterclockwise, the distribution P_{D, a_1, \dots, a_4} viewed as a distribution on pairs of SLE paths that join these four boundary points has the following properties:

- They are conformally invariant. That is, if Φ is a conformal transformation, then the law of the image of P_{D, a_1, \dots, a_4} under Φ is $P_{\Phi(D), \Phi(a_1), \dots, \Phi(a_4)}$. This shows in particular that the probability that a_1 hooks up with a_4 is in fact a function $H(x)$ of the cross-ratio x of a_1, \dots, a_4 .
- For any given a_i , if one discovers the entire strand γ_i that emanates from a_i (and therefore its endpoint a_j), then the conditional distribution of the other remaining strand is just an SLE_κ joining the two remaining marked point in $D \setminus \gamma_i$.

Hence, if one conditions P_{D, a_1, \dots, a_4} on the event that a_1 is connected to a_2 (and therefore that a_3 is connected to a_4), one gets a distribution on pairs of paths (γ, γ') joining a_1 to a_2 and a_3 to a_4 respectively, such that (i) conditionally on γ , the law of γ' is that of SLE_κ in $D \setminus \gamma$, and (ii) conditionally on γ' , the law of γ is that of SLE_κ in $D \setminus \gamma'$. It is possible to see (and this has been done using several methods) that the law on pairs (γ, γ') is uniquely characterized by this last property (this is the resampling property of bichordal SLE as studied and used in [32, 33, 34, 35]).

This explains why P_{D, a_1, \dots, a_4} will be fully determined once one knows the hook-up probability function $H(x)$ and in fact, that it suffices to know the value of $H(x_0)$ for one single value $x_0 \in (0, 1)$ in order to deduce the entire function H . Indeed, if one knows P_{D, a_1, \dots, a_4} for one given choice of (D, a_1, \dots, a_4) , then H is determined because the hook-up probability evolves as a martingale when one lets one strand evolve, we know the law of the evolution of this strand, and the boundary values at 0 and 1. As explained in [37], one can also view this as the evolution of $\text{SLE}_\kappa(\kappa - 6)$ conditionally on part of its time-reversal.

Considerations of this type are in fact included in some form in the papers cited above that introduce and study commutation relations for SLE paths and their consequences (note that these

in fact actually study a somewhat more general class of questions – in the present setup, we for instance already know from the construction that our commuting strands will eventually hook-up and create one or two loops, which is a non-trivial feature). Then, it follows from these arguments that H is of the form

$$(4.1) \quad H(x) = \frac{Z(x)}{Z(x) + \theta Z(1-x)}$$

for some positive θ , where

$$Z(x) := x^{2/\kappa}(1-x)^{1-6/\kappa}f(x),$$

and where here and in the remainder of this paper, f will denote the hypergeometric function

$$f(x) := F(4/\kappa, 1 - 4/\kappa, 8/\kappa; x)$$

(see for instance [11, Section 4] or [3, Section 8] about “4-SLE”). Recall (see the short Appendix B where we will briefly recall basics about hypergeometric functions) that $f(0) = 1$ and note that since $8/\kappa - (4/\kappa + 1 - 4/\kappa) = 8/\kappa - 1 > 0$, this function f is continuous at 1 with

$$f(1) = \frac{\Gamma(8/\kappa)\Gamma(8/\kappa - 1)}{\Gamma(4/\kappa)\Gamma(12/\kappa - 1)}.$$

Another way to phrase/interpret this in the previous setup is that the above-mentioned papers describe the law of the bichordal SLEs, which are the conditional distributions of P_{D, a_1, \dots, a_4} given one hook-up event (or given the other one), but not the actual probability of these hook-up events. In other words, in order to determine the function H , it only remains to identify the value of θ in terms of κ . Note that knowing the value of θ is equivalent to knowing the connection probability for a conformal square (as it is equal to $1/(1 + \theta)$).

Note that that as $x \rightarrow 0$,

$$Z(x) \sim x^{2/\kappa} \quad \text{and} \quad Z(1-x) \sim x^{1-6/\kappa}f(1).$$

Hence, because $1 - 6/\kappa < 2/\kappa$, it follows that $H(x) \sim x^{8/\kappa-1}/(\theta f(1))$ as $x \rightarrow 0$. In other words,

$$(4.2) \quad \theta^{-1} = f(1) \lim_{x \rightarrow 0} x^{1-8/\kappa} H(x).$$

The strategy of our proof of Theorem 1 i.e., of the fact that $\theta = -2 \cos(4\pi/\kappa)$, will be to determine the right-hand side of (4.2), which therefore also gives the value of θ . In other words, we will in fact estimate precisely the asymptotics of the hook-up probability in very thin conformal rectangles. We have just argued that $H(x)$ decays like some constant times $x^{8/\kappa-1}$ as $x \rightarrow 0$, and our goal will be to determine the value of this constant.

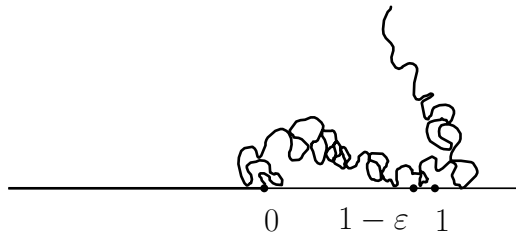
5. OUTLINE OF THE PROOFS OF THEOREM 1

We now describe the main steps of the proof of Theorem 1 in the cases $\kappa \in (4, 8)$ and $\kappa \in (8/3, 4]$ separately. We will defer the proofs of the more computational lemmas to the next sections, in order to highlight here the arguments that reduce the proof of Theorem 1 to concrete computations involving SLE and Bessel processes.

5.1. Case of non-simple CLEs. Let us start with the case of CLE_κ when $\kappa \in (4, 8)$. Consider a conditioned CLE_κ in the upper half-plane with one wired boundary arc on \mathbf{R}_- as described before. Recall that the law of this conditioned CLE_κ can be sampled from using the following steps:

- Sample an $\text{SLE}_\kappa \gamma$ from 0 to ∞ in order to complete the partially discovered loop that runs on \mathbf{R}_- , and then

- Sample independent CLE_κ 's in the remaining connected components that are “outside” of this loop.


 FIGURE 8. Sketch of the event $D(\varepsilon)$

We now fix $\varepsilon > 0$ very small and define the event $D(\varepsilon)$ that $\gamma \cap [1 - \varepsilon, 1] \neq \emptyset$ (see Figure 8). The probability of $D(\varepsilon)$ can be explicitly computed (it is in fact the formula that was already used by Schramm [44] in his argument mentioned at the beginning of the introduction); it is a generalization of Cardy's formula for SLE_κ almost identical to that determined in [27] – see for instance [43, Lemma 6.6] or Section 3 of [57]):

$$\mathbf{P}[D(\varepsilon)] = \frac{\int_{1/\varepsilon}^{\infty} y^{-4/\kappa} (1+y)^{-4/\kappa} dy}{\int_0^{\infty} y^{-4/\kappa} (1+y)^{-4/\kappa} dy}.$$

Clearly, as $\varepsilon \rightarrow 0$,

$$(5.1) \quad \mathbf{P}[D(\varepsilon)] \sim \frac{\varepsilon^{8/\kappa-1}}{(8/\kappa-1) \int_0^{\infty} y^{-4/\kappa} (1+y)^{-4/\kappa} dy} \sim \frac{\Gamma(4/\kappa)}{\Gamma(1-4/\kappa)\Gamma(8/\kappa)} \varepsilon^{8/\kappa-1}.$$

The idea is now to evaluate the asymptotic behavior of $\mathbf{P}[D(\varepsilon)]$ as $\varepsilon \rightarrow 0$ using a different two-step procedure that will involve hook-up probabilities: We first explore the conditioned CLE_κ by discovering progressively the loops from left to right (each loop being traced in clockwise direction) that are attached to the segment $[1 - \varepsilon, 1]$ and we stop at the first time τ (if it exists) at which the cross-ratio between $(\infty, 0, w_\tau, o_\tau)$ reaches $x = \varepsilon^{7/8}$, where here and in the following lines, w_t will denote the tip of the exploration at time t and o_t the other end of the portion the loop that has been traced so far. We call $B(\varepsilon)$ the event that such a time exists.


 FIGURE 9. Sketch of the event $B(\varepsilon)$

The following lemma will enable us to relate the asymptotic behavior of $H(x)$ as $x \rightarrow 0$ to that of $\mathbf{P}[B(\varepsilon)]$:

Lemma 2. *One has $D(\varepsilon) \subset B(\varepsilon)$. Furthermore, if $D'(\varepsilon)$ denotes the event that the partially explored loop at time τ (in the definition of $B(\varepsilon)$) does in fact correspond to a portion of γ , then the conditional probability of $D'(\varepsilon)$ given $D(\varepsilon)$ tends to 1 as $\varepsilon \rightarrow 0$.*

Proof. Recall that we are working with a CLE_κ in the upper half-plane with wired boundary conditions on \mathbf{R}_- , which consists of an SLE_κ that we will denote by γ and a family of further loops to the right of it. When one explores the CLE_κ loops of such a wired CLE_κ that touch $[1 - \varepsilon, 1]$ starting from $1 - \varepsilon$, and tracing them in the clockwise direction one after the other, then in the configuration where γ intersects this segment (i.e., when $D(\varepsilon)$ holds), at some point, one has to trace an arc of the loop that γ is part of, and that connects a point in $[1 - \varepsilon, 1]$ to a point that lies in \mathbf{R}_- , as depicted in Figure 10. Just before this time, the cross-ratio between the four points $(\infty, 0, w_t, o_t)$ tends to 1 because w_t approaches \mathbf{R}_- , which implies that it did reach $\varepsilon^{7/8}$ beforehand. Hence, $D(\varepsilon)$ is indeed a subset of $B(\varepsilon)$.

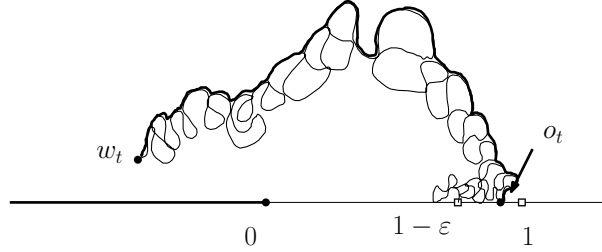


FIGURE 10. The two bold boundary parts get very close

Suppose now that we are in the case where $B(\varepsilon)$ holds but not $D'(\varepsilon)$. Then, at the time σ at which one has completed the loop that one was tracing at time τ , the conditional distribution in the remaining to be explored domain will be again a CLE_κ with just one wired boundary arc on \mathbf{R}_- . The conditional probability that $D(\varepsilon)$ still holds will therefore be smaller than the unconditional probability that $D(\varepsilon)$ holds because the cross-ratio between the four points $(o_\sigma, 1, \infty, 0)$ at that time is necessarily smaller than ε (see Figure 11). In other words,

$$\mathbf{P}[D(\varepsilon) \setminus D'(\varepsilon)] \leq \mathbf{P}[B(\varepsilon)] \times \mathbf{P}[D(\varepsilon)].$$

It finally remains to note that $\mathbf{P}[B(\varepsilon)] \rightarrow 0$ as $\varepsilon \rightarrow 0$, as a consequence of the bound $\mathbf{P}[B(\varepsilon)] \times$



FIGURE 11. At such a time, the conditional probability that $D(\varepsilon)$ holds is smaller than the unconditional probability.

$H(\varepsilon^{7/8}) \leq \mathbf{P}[D(\varepsilon)]$ and our previous estimates of H and $\mathbf{P}[D(\varepsilon)]$. Hence, the conditional probability of $D'(\varepsilon)$ given $D(\varepsilon)$ tends to 1 as $\varepsilon \rightarrow 0$, which concludes the proof. \square

The previous lemma implies in particular that

$$(5.2) \quad \frac{P[D(\varepsilon)]}{P[B(\varepsilon)]} = \mathbf{P}[D(\varepsilon)|B(\varepsilon)] \sim \mathbf{P}[D'(\varepsilon)|B(\varepsilon)] = H(\varepsilon^{7/8})$$

as $\varepsilon \rightarrow 0$. The proof of Theorem 1 for $\kappa \in (4, 8)$ will then be complete if we prove the following estimate:

Lemma 3. *As $\varepsilon \rightarrow 0$,*

$$\mathbf{P}[B(\varepsilon)] \sim \frac{\Gamma(4/\kappa)}{\Gamma(2 - 8/\kappa)\Gamma(12/\kappa - 1)} \times (\varepsilon^{1/8})^{8/\kappa - 1}.$$

Indeed, combining this lemma with (5.1) and (5.2) shows that as $x = \varepsilon^{7/8} \rightarrow 0$,

$$H(x) \sim \frac{\mathbf{P}[D(\varepsilon)]}{\mathbf{P}[B(\varepsilon)]} \sim \frac{\varepsilon^{8/\kappa - 1}}{(\varepsilon^{1/8})^{8/\kappa - 1}} \times \frac{\Gamma(2 - 8/\kappa)\Gamma(12/\kappa - 1)}{\Gamma(1 - 4/\kappa)\Gamma(8/\kappa)} \sim \frac{\Gamma(2 - 8/\kappa)\Gamma(12/\kappa - 1)x^{8/\kappa - 1}}{\Gamma(1 - 4/\kappa)\Gamma(8/\kappa)}.$$

Combining this with (4.2), we see that:

$$\theta^{-1} = f(1) \frac{\Gamma(2 - 8/\kappa)\Gamma(12/\kappa - 1)}{\Gamma(1 - 4/\kappa)\Gamma(8/\kappa)} = \frac{\Gamma(2 - 8/\kappa)\Gamma(8/\kappa - 1)}{\Gamma(1 - 4/\kappa)\Gamma(4/\kappa)} = \frac{\sin(4\pi/\kappa)}{\sin(\pi(8/\kappa - 1))} = \frac{-1}{2\cos(4\pi/\kappa)}$$

(recalling that $\Gamma(1 - z)\Gamma(z) = \pi/\sin(\pi z)$).

The proof of Lemma 3 will be presented in Section 6.

5.2. Case of simple CLEs. In the case where $\kappa \in (8/3, 4]$, we are also going to estimate the asymptotic behavior of $H(x)$ as $x \rightarrow 0$, but we need a somewhat different strategy because SLE_κ paths do not hit boundary intervals anymore. The similarity with the case $\kappa \in (4, 8)$ is that we will again estimate the asymptotic behavior of $H(x)$ as $x \rightarrow 0$ by estimating the asymptotic behavior of the probability of another event $C(x)$, for which we show that $P[C(x)] \sim H(x)$.

We consider a CLE_κ in the upper half-plane, with boundary conditions that are respectively wired, free, wired and free on \mathbf{R}_- , $[0, 1 - \varepsilon]$, $[1 - \varepsilon, 1]$ and $[1, \infty)$. So, we have four strands starting at ∞ , 0 , $1 - \varepsilon$ and 1 , and $H(\varepsilon)$ is the probability that the strand starting from 0 hooks up with the one starting from $1 - \varepsilon$.

The first key observation is the following:

Lemma 4. *As $\varepsilon \rightarrow 0$, the probability $H(\varepsilon)$ is in fact equivalent to the probability of the event $C(\varepsilon)$ that an SLE_κ path from 0 to ∞ intersects an independent one-sided restriction measure of exponent $\alpha = (6 - \kappa)/(2\kappa)$ from $1 - \varepsilon$ to 1 (see Figure 12).*

The proof of this lemma will combine the construction of CLE_κ via Brownian loop-soup clusters (and the decomposition after partial explorations recently derived in [41]), with the fact that when a loop-soup cluster gets close to a boundary point, it typically does so via a chain of Brownian loops and not because of one single macroscopic Brownian loop gets close to that point.

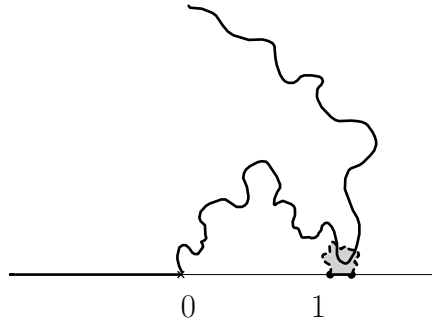


FIGURE 12. Sketch of the event $C(\varepsilon)$

Proof. Let us consider a $\text{CLE}_\kappa \mathcal{C}$ for $\kappa \in (8/3, 4]$ in the unit disk (with free boundary conditions) that is generated by a Brownian loop-soup \mathcal{L} , and let us first focus on the configurations in the η -neighborhoods of the two boundary points 1 and -1 .

We start a Markovian exploration near -1 stopped at some stopping time, where one is currently tracing a CLE_κ loop and that has the property that it is stopped before exiting the η -neighborhood of -1 . Let us call ∂_{-1} the portion of loop that one started tracing. Then, the conditional distribution of the Brownian loop-soup in the remaining domain D_{-1} can be decomposed as follows (see [41]): the union of the Brownian loops that touch ∂_{-1} form a restriction sample \mathcal{R}_{-1} of exponent $\alpha = (6 - \kappa)/(2\kappa)$ attached to ∂_{-1} the remaining to be discovered region, and the other Brownian loops form an independent Brownian loop-soup in this domain. This loop-soup forms a $\text{CLE}_\kappa \mathcal{C}_{-1}$ in the remaining to be discovered domain (looking at the outermost boundaries of loop-soup clusters); the outer boundary of the union of the restriction measure with the loops of \mathcal{C}_{-1} that it intersects forms the SLE curve that correspond to the end of the loop of the original $\text{CLE}_\kappa \mathcal{C}$ that contains ∂_{-1} . Another important property of this description is that the CLE_κ exploration is independent of the set of Brownian loops in \mathcal{L} that do not intersect the η -neighborhood of -1 .

After having sampled this first exploration, we launch another exploration from the boundary point 1, but only after changing \mathcal{L} (and therefore the CLE_κ) that one is exploring, by resampling the collection of Brownian loops that intersect the η -neighborhood of -1 . In this way, we are actually discovering a loop-soup that is totally independent from the first exploration, and from the restriction sample \mathcal{R}_{-1} that one discovered there. We then stop this second exploration at some stopping time, before it exits the η -neighborhood of 1. Note that we allow the possibility to choose the stopping time using some information from the first exploration (i.e., we perform this second exploration near 1 after having already discovered the first one).

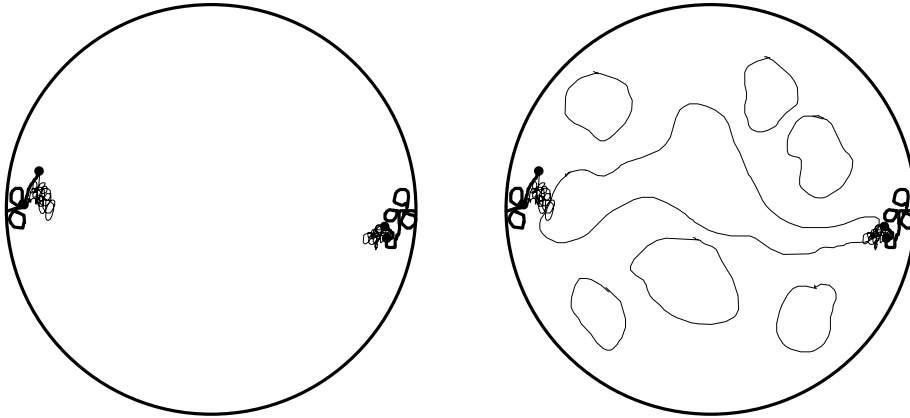


FIGURE 13. The two explorations with the Brownian loops that they discovered (left). The two wired boundaries are part of the same loop if a CLE_κ loops intersects both restriction samples (up to a small probability).

One important very simple a priori estimate is that the probability that there exists a Brownian loop in the loop-soup that intersects both the η -neighborhoods of 1 and -1 decays like a constant times η^4 as $\eta \rightarrow 0$. Hence, we see that one can couple the following two constructions:

- Consider one loop-soup, and explore it first near 1, and then near -1 in such a way that the two explorations do not exit the η -neighborhood of their starting points,

- Consider as before two explorations of independent loop-soups, one near 1 and the other near -1 , following the same “exploration rules” (in order to define the stopping times) as the first one,

in such a way that, up to an event of probability $O(\eta^4)$, the two pairs of explorations coincide, and the collection of Brownian loops that they did respectively discover coincide as well.

In the first procedure, the undiscovered Brownian loops (that touch none of the two portions of loops) will form an independent loop-soup (and therefore a CLE_κ) in the remaining to be discovered domain (or for our purposes, it suffices to say that they can be coupled to such a configuration with a probability $1 - O(\eta^4)$). It therefore follows that the two discovered portions of CLE_κ loops will hook-up into a single CLE_κ loop if and only if there exists a Brownian loop touching the first one (near -1) that intersects a CLE_κ loop in the remaining to be discovered domain that intersect a Brownian loop that touches the second discovered portion (near 1), up to an event of probability $O(\eta^4)$.

To conclude, we remark that with for some large but fixed m , with positive probability bounded from below independently of ε , the two explorations near 1 and -1 stay in the $m\sqrt{\varepsilon}$ neighborhood of -1 and 1 respectively and create a configuration such that the cross-ratio between the two discovered portion of loops in the remaining to be discovered domain is exactly equal to ε (we call this event $K(\varepsilon)$). Conditionally on $K(\varepsilon)$, the hook-up probability between these two boundaries is exactly $H(\varepsilon)$, which is of the order of some constant times $\varepsilon^{8/\kappa-1}$ as $\varepsilon \rightarrow 0$.

On the other hand, we have just argued that (taking $\eta = m\sqrt{\varepsilon}$), up to an event of probability $O(\varepsilon^2)$, the hook-up will occur if and only if a restriction sample attached to one wired portion intersects an independent SLE_κ joining the extremities of the other one, which is an event of conditional probability $\mathbf{P}[C(\varepsilon)]$ on the event $K(\varepsilon)$. Since $8/\kappa - 1 < 2$, we can indeed conclude that $H(\varepsilon)$ is asymptotically equivalent to $\mathbf{P}[C(\varepsilon)]$. \square

The proof of Theorem 1 will then be complete if we estimate $\mathbf{P}[C(\varepsilon)]$ as follows:

Lemma 5. As $\varepsilon \rightarrow 0$,

$$\mathbf{P}[C(\varepsilon)] \sim \frac{\varepsilon^{8/\kappa-1}}{f(1) \times (-2 \cos(4\pi/\kappa))}.$$

Indeed, combining this lemma with (4.2) and Lemma 4 then shows that indeed $\theta = -2 \cos(4\pi/\kappa)$. The proof of Lemma 5 is an exercise about SLE and hypergeometric functions that is performed in Section 7.

6. PROOF OF LEMMA 3

In the present section, we will prove Lemma 3, which will complete the proof of Theorem 1 for $\kappa \in (4, 8)$.

Let us first derive another result that will be useful in our proof of Lemma 3. We will derive here properties of the usual CLE_κ (with no wired boundary part) for $\kappa \in (4, 8)$. Note that $8/\kappa \in (1, 2)$ for $\kappa \in (4, 8)$ (we will implicitly and repeatedly use this fact in the following arguments). Consider an $\text{SLE}_\kappa(\kappa - 6)$ in \mathbf{H} starting from w , with initial marked point $o \geq w$, and targeted at ∞ . Recall that the law of the driving function W_t of this Loewner chain can be sampled from using the following two steps:

- Sample a reflected Bessel process X with dimension $d = 3 - 8/\kappa \in (1, 2)$ started from $(o - w)/\sqrt{\kappa}$ (at the end of the day, the process $\sqrt{\kappa}X_t$ will be equal to $O_t - W_t$, the difference between the force point and the driving process).

- Set

$$O_t = o + \int_0^t \frac{2}{\sqrt{\kappa} X_s} ds \quad \text{and} \quad W_t = O_t - \sqrt{\kappa} X_t.$$

Note that the image under g_t of the left-most point o_t on $[o, \infty)$ that has not been swallowed by the Loewner chain before time t is equal to O_t . For each $b > 0$, let T_b be the first time t that b is swallowed by the Loewner chain. As the Bessel process dimension d is strictly between 1 and 2, we have that $T_b < \infty$ a.s.

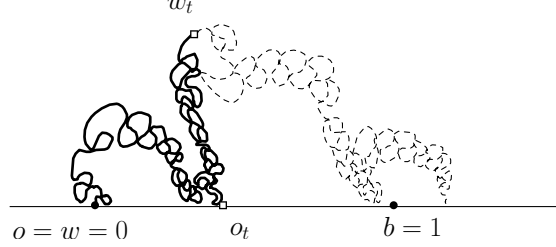


FIGURE 14. Sketch of the $\text{SLE}_\kappa(\kappa - 6)$ where the initial locations of the driving function and force point are given by the origin.

We will also use the local time at the origin of the process $O - W$, which is a multiple of the local time at the origin of the Bessel process X . More precisely, we define this local time as

$$\ell_t := \lim_{\varepsilon \rightarrow 0} \varepsilon^{8/\kappa - 1} N_t^{0 \rightarrow \varepsilon}$$

where $N_t^{0 \rightarrow \varepsilon}$ denotes the number of upcrossings from 0 to ε by $O - W$ before time t . Let τ_ε be the first time that $O - W$ hits ε . Due to our particular normalization for the definition of the local time, we have that the expected value of ℓ at time τ_ε is exactly $\varepsilon^{8/\kappa - 1}$ when $w = o = 0$.

The goal of this section is to prove the following fact, that can be viewed as a statement about the Bessel flow.

Lemma 6. *Suppose that $w = o = 0$. Then*

$$\mathbf{E}[\ell_{T_1}] = \frac{\Gamma(4/\kappa)}{\Gamma(2 - 8/\kappa)\Gamma(12/\kappa - 1)}.$$

Proof. Let us define the more general function $L(a, b)$ to be the expected value of ℓ_{T_b} when the process is started from $w = -a$, $o = 0$. By scaling, we have that

$$L(a, b) = b^{8/\kappa - 1} L(a/b, 1).$$

Let us define the function U on $[1, \infty)$ so that $L(a/b, 1) = U(a/b + 1)$. With this notation, our goal is therefore to determine

$$u_1 := L(0, 1) = U(1).$$

When the Bessel process evolves away from 0, the local time at zero does not change. Hence, $L(O_t - W_t, g_t(1) - O_t)$ is a local martingale up to the first hitting time of 0 by $O - W$. This implies (using the standard arguments for SLE martingales) that U is smooth on $(1, \infty)$ and satisfies

$$(1 - x)xU''(x) + \left(\frac{4}{\kappa} + \left(\frac{4}{\kappa} - 2\right)x\right)U'(x) + \frac{4}{\kappa}\left(\frac{8}{\kappa} - 1\right)U(x) = 0$$

with boundary conditions

$$U(1) = u_1 \quad \text{and} \quad \lim_{x \rightarrow \infty} U(x) = 0.$$

In other words (see Appendix B), the function U is equal to

$$U(x) = u_1 x^{-4/\kappa} \frac{F(4/\kappa, 1, 12/\kappa; 1/x)}{F(4/\kappa, 1, 12/\kappa; 1)}$$

(note that the ODE is exactly the equation (B.1) for the coefficients $a = c = 4/\kappa$, $b = 1 - 8/\kappa$, so that U is a multiple of the function h_1 defined in the Appendix B).

Our goal in the next paragraph is to show that

$$(6.1) \quad L(0, 1) = L(h, 1) + h^{8/\kappa-1} + o(h^{8/\kappa-1}) \quad \text{as } h \rightarrow 0.$$

This will then enable us to identify u_1 . Let us define for all positive h ,

$$Y_h := \mathbf{1}_{\tau_h < T_1} (g_{\tau_h}(1) - O_{\tau_h}).$$

Note that by the monotonicity properties of the Bessel flow, $Y_h \leq 1$ when one starts with $o = w = 0$. Using the Markov property at $\min(T_1, \tau_h)$ and the additivity of the local time we see that

$$L(0, 1) = \mathbf{E}[\ell_{T_1}] = \mathbf{E}[\mathbf{1}_{T_1 < \tau_h} \ell_{T_1}] + \mathbf{E}[\mathbf{1}_{\tau_h < T_1} \ell_{\tau_h}] + \mathbf{E}[L(h, Y_h)]$$

(note that by definition $L(h, Y_h) = \mathbf{1}_{\tau_h < T_1} L(h, Y_h)$). Using the scaling property, the fact that the probability that $T_1/h < \tau_1$ tends to 0 as $h \rightarrow 0$, and the fact that $\mathbf{E}[\ell_{\tau_h}] = h^{8/\kappa-1}$ (this is where we use our actual normalization in the definition of the local time), we get that

$$\begin{aligned} & |\mathbf{E}[\mathbf{1}_{T_1 < \tau_h} \ell_{T_1}] + \mathbf{E}[\mathbf{1}_{\tau_h < T_1} \ell_{\tau_h}] - h^{8/\kappa-1}| \\ &= \mathbf{E}[\mathbf{1}_{T_1 < \tau_h} (\ell_{\tau_h} - \ell_{T_1})] \leq \mathbf{E}[\mathbf{1}_{T_1 < \tau_h} \ell_{\tau_h}] = h^{8/\kappa-1} \mathbf{E}[\mathbf{1}_{T_1/h < \tau_1} \ell_{\tau_1}] = o(h^{8/\kappa-1}). \end{aligned}$$

It therefore remains to estimate $\mathbf{E}[L(h, 1) - L(h, Y_h)]$. Note that once we condition on $Y_h = y$, we can use the same flow to couple the realizations that lead to $L(h, 1)$ and $L(h, y)$ (defined as expected values of local times). The difference between the two quantities will therefore be due to the configurations in this coupling where the $\text{SLE}_\kappa(\kappa - 6)$ hits the interval $[y, 1]$ for which one then counts the local time accumulated after that time but before T_1 . This is an event that has probability bounded by a constant times $(1 - y)^\beta$ for some $\beta > 0$. We remark that it is in fact known that $\beta = (\kappa - 4)^2/(2\kappa) > 0$, see [39, Theorem 1.8]. For what follows, we will only use that $\beta > 0$ and in fact not need such a precise bound. Hence, for some constant $C > 0$ and all $h > 0$ and $y < 1$, we have that

$$|L(h, 1) - L(h, y)| \leq C(1 - y)^{\beta+8/\kappa-1}$$

and therefore

$$\begin{aligned} & |\mathbf{E}[L(h, Y_h)] - L(h, 1)| \\ & \leq C \mathbf{E}[(1 - Y_h)^{\beta+8/\kappa-1}] \leq C' (\mathbf{E}[|O_{\tau_h}|^{\beta+8/\kappa-1}] + \mathbf{E}[\mathbf{1}_{\tau_h < T_1} |1 - g_{\tau_h}(1)|^{\beta+8/\kappa-1}]) \end{aligned}$$

where $C' > 0$ is a constant. By scaling, we note that

$$\mathbf{E}[|O_{\tau_h}|^{\beta+8/\kappa-1}] = h^{8/\kappa-1+\beta} \mathbf{E}[|O_{\tau_1}|^{\beta+8/\kappa-1}].$$

On the other hand, it is also easy to see that $\mathbf{E}[\mathbf{1}_{\tau_h < T_1} |1 - g_{\tau_h}(1)|^{\beta+8/\kappa-1}]$ decays to 0 faster than $h^{8/\kappa-1+\beta}$ as well: Typically, τ_h will be of order h^2 (because of scaling), so that $g_{\tau_h}(1) - 1$ will be of order h^2 as well. Simple estimates about the probability that τ_h is exceptionally large or W fluctuates exceptionally on a small time interval allows us to conclude. Putting the pieces together, we then get indeed (6.1).

This is now enough to pin down the exact value of u_1 . Indeed (6.1) implies that when $h \rightarrow 0$,

$$U(1) - U(1 + h) \sim h^{8/\kappa-1}.$$

Since we know that in the right neighborhood of 1, U has to be a linear combination of $F(4/\kappa, 1 - 8/\kappa, 2 - 8/\kappa; 1 - x)$ and of $(x - 1)^{8/\kappa} F(0, 12/\kappa - 1, 8/\kappa; 1 - x)$, by looking at the expansion near 1, we conclude that

$$U(x) = u_1 F(4/\kappa, 1 - 8/\kappa, 2 - 8/\kappa; 1 - x) - (x - 1)^{8/\kappa} F(0, 12/\kappa - 1, 8/\kappa; 1 - x).$$

On the other hand, we have seen that U is also a multiple of the function $x^{-4/\kappa} F(4/\kappa, 1, 12/\kappa; 1/x)$, and by comparing this with the connection formula (B.3) that relates these three hypergeometric functions, we see that $-u_1$ is the ratio of the two coefficients on the right-hand side of (B.3) for the appropriate choice of $a = 4/\kappa$, $b = 1 - 8/\kappa$ and $c = 4/\kappa$, and we obtain that

$$u_1 = \frac{-\Gamma(8/\kappa - 1)\Gamma(4/\kappa)}{\Gamma(8/\kappa)\Gamma(12/\kappa - 1)\Gamma(1 - 8/\kappa)}$$

which proves the claim. \square

This lemma will be used in the proof of Lemma 3 via the following corollary:

Corollary 7. *Assume that $w = b = 0$. Then, as $y \rightarrow 0$,*

$$\mathbf{P}[\tau_{y^{3/4}} < T_y] \sim (y^{1/4})^{8/\kappa-1} \times \frac{\Gamma(4/\kappa)}{\Gamma(2 - 8/\kappa)\Gamma(12/\kappa - 1)}.$$

Proof. The properties of Poisson point processes, our normalization of the Bessel local time and scaling show that the expected value of the number \mathcal{N} of excursions of $O - W$ that reach level $y^{3/4}$ before time T_y is

$$\mathbf{E}[\mathcal{N}] = (y^{3/4})^{-(8/\kappa-1)} \times \mathbf{E}[\ell_{T_y}] = (y^{1/4})^{8/\kappa-1} \times \mathbf{E}[\ell_{T_1}].$$

It is also easy to see, using similar arguments as above, that $\mathbf{P}[\mathcal{N} \geq n] \leq \mathbf{P}[\mathcal{N} \geq 1]^n$, so that in fact,

$$\mathbf{E}[\mathcal{N} \mathbf{1}_{\mathcal{N} \geq 2}] = \mathbf{P}[\mathcal{N} \geq 2] + \sum_{n \geq 2} \mathbf{P}[\mathcal{N} \geq n] \leq 4\mathbf{P}[\mathcal{N} \geq 1]^2$$

for all small enough y . It follows that as $y \rightarrow 0$,

$$\mathbf{P}[\tau_{y^{3/4}} < T_y] = \mathbf{P}[\mathcal{N} \geq 1] \sim \mathbf{P}[\mathcal{N} = 1] \sim \mathbf{E}[\mathcal{N}] \sim \frac{\Gamma(4/\kappa)}{\Gamma(2 - 8/\kappa)\Gamma(12/\kappa - 1)} \times (y^{1/4})^{8/\kappa-1}.$$

\square

It now remains to deduce Lemma 3 from Corollary 7. Instead of working with the CLE_κ in \mathbf{H} with wired boundary on \mathbf{R}_- and using the additional marked points at $1 - \varepsilon$ and 1, we will instead work in the unit disk \mathbf{D} and choose the four points a_ε , \bar{a}_ε , $-a_\varepsilon$ and $-\bar{a}_\varepsilon$ on the unit circle, where a_ε is chosen very close to 1 so that the cross-ratio between these four points is exactly ε . Note that as $\varepsilon \rightarrow 0$, $|a_\varepsilon - 1|$ is of the order of $\sqrt{\varepsilon}$. These four points define two small boundary arcs ∂_ε and $-\partial_\varepsilon$, respectively near 1 and -1 .

By conformal invariance, the event $B(\varepsilon)$ becomes the event $B'(\varepsilon)$ that, if one looks at the CLE with wired boundary condition on $-\partial_\varepsilon$ and explores the loops (of this wired CLE) attached to ∂_ε in their order of appearance starting from \bar{a}_ε , one finds a time at which the cross-ratio between $(-a_\varepsilon, -\bar{a}_\varepsilon, w_t, o_t)$ reaches $\varepsilon^{7/8}$ (note that typically, this will occur for loops attached to ∂_ε that are of size of the order of $\varepsilon^{3/8}$).

In order to apply our previous estimates for non-conditioned CLE's, we first sample the SLE γ that joins the end-points of $-\partial_\varepsilon$. By the same $8/\kappa - 1$ boundary exponent for SLE, we know that the probability that this SLE has diameter greater than $\varepsilon^{1/4}$ is bounded by a constant times $(\varepsilon^{1/4})^{8/\kappa-1}$. On the event that the diameter of γ is smaller than $\varepsilon^{1/4}$ (which has probability very close to 1 by the previous estimate), we now look at the CLE in the complement of this small SLE:

We can first map the connected component of the complement of this curve which has ∂_ε on its boundary back to the unit disk in such a way that $a_\varepsilon, \bar{a}_\varepsilon$ are fixed and (say) the two extremal points on $\gamma \cap \partial \mathbf{D}$ are mapped onto symmetric points on the real axis – this defines a conformal map φ that (by standard distortion estimates) is uniformly very close to the identity map (the derivative of this map is uniformly close to 1 on the right-half of the unit disk) in the neighborhood of 1.

We can also now discover the CLE in this disk, by using the $\text{SLE}_\kappa(\kappa - 6)$ exploration in the upper half-plane (which defines a process W and O) and mapping it back onto the disk via the conformal map from \mathbf{H} onto \mathbf{D} that maps ∞ to -1 and is normalized in the neighborhood of ∞ . Then, distortion estimates for conformal maps show also that the cross-ratio between the four points $(-a_\varepsilon, -\bar{a}_\varepsilon, w_t, o_t)$ in the original domain is very close to $\sqrt{\varepsilon}$ times $W_t - O_t$ (i.e., the ratio between the two is uniformly close to 1, as long as the $\text{SLE}_\kappa(\kappa - 6)$ stays in the neighborhood of its starting point).

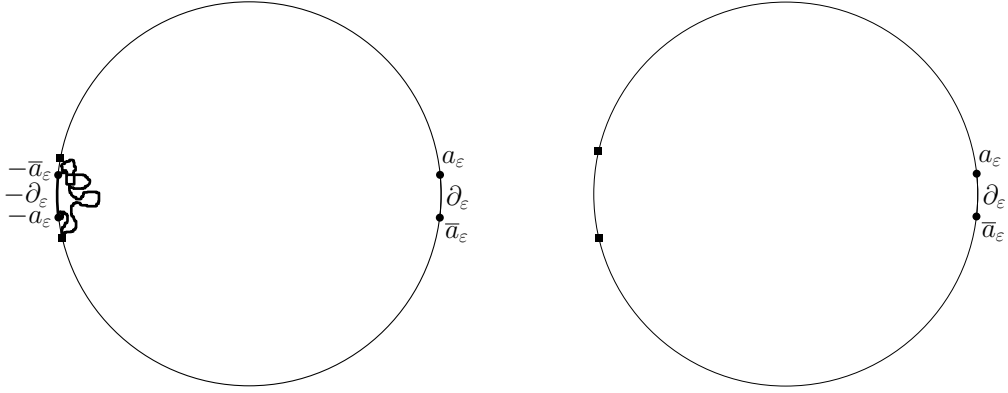


FIGURE 15. The SLE from $-a_\varepsilon$ to $-\bar{a}_\varepsilon$ in the wired CLE_κ . The uniformizing map onto the unit disk is very close to the identity in the right-hand side of the disk, and therefore also near ∂_ε .

This shows that the probability of $B(\varepsilon)$ is up to an error of order $(\varepsilon^{1/4})^{8/\kappa-1}$ asymptotic to the probability that $O - W$ hits $\varepsilon^{3/8}$ before swallowing ∂_ε i.e., to $\mathbf{P}[\tau_{y^{3/4}} < T_y]$ in Corollary 7 for $y = \sqrt{\varepsilon}$. Hence, we conclude that indeed,

$$\mathbf{P}[B(\varepsilon)] \sim (\varepsilon^{1/8})^{8/\kappa-1} \times \frac{\Gamma(4/\kappa)}{\Gamma(2 - 8/\kappa)\Gamma(12/\kappa - 1)},$$

which concludes the proof of Lemma 3.

7. PROOF OF LEMMA 5

In the present section, we will prove Lemma 5 that will complete the proof of Theorem 1 for $\kappa \in (8/3, 4]$.

Let us use the standard (half-plane capacity) parameterization and notation for the SLE_κ denoted by γ driven by $W_t = \sqrt{\kappa}\beta_t$ and let g_t be the uniformizing conformal map from $\mathbf{H} \setminus \gamma[0, t]$ onto \mathbf{H} with the normalization $g_t(z) - z \rightarrow 0$ as $z \rightarrow \infty$, so that $\partial_t g_t(z) = 2/(g_t(z) - W_t)$.

We denote by \mathcal{R} the independent one-sided restriction sample attached to $[1 - \varepsilon, 1]$. Our goal is to estimate the probability of the event $C(\varepsilon)$ that γ intersects \mathcal{R} . Let us define the function Q on $(0, 1)$ by

$$Q(1-\varepsilon) := 1 - \mathbf{P}[C(\varepsilon)] = \mathbf{P}[\gamma \cap \mathcal{R} = \emptyset] = \lim_{t \rightarrow \infty} \mathbf{P}[\gamma[0, t] \cap \mathcal{R} = \emptyset] = \lim_{t \rightarrow \infty} \mathbf{E} \left[\left(\frac{\varepsilon^2 g'_t(1 - \varepsilon) g'_t(1)}{(g_t(1) - g_t(1 - \varepsilon))^2} \right)^\alpha \right].$$

If we write $I_t = g_t(1)$ and $V_t = g_t(1 - \varepsilon)$, then we note that

$$Q((V_t - W_t)/(I_t - W_t)) \times \left(\frac{\varepsilon^2 g'_t(1 - \varepsilon) g'_t(1)}{(g_t(1) - g_t(1 - \varepsilon))^2} \right)^\alpha$$

is a bounded martingale, and we deduce, using the standard machinery that Q is smooth and is a solution to the ODE

$$\frac{\kappa}{2} x^2 (x - 1) Q''(x) + ((\kappa - 2)x - 2)x Q'(x) - 2\alpha(x - 1)Q(x) = 0$$

on $(0, 1)$ with boundary conditions $Q(0) = 0$ and $Q(1) = 1$. In fact, if we write $Q(x) = x^{2/\kappa} A(x)$, then A solves the following hypergeometric differential equation

$$(7.1) \quad x(1 - x)A'' + (-2x + 8/\kappa)A' + (16/\kappa^2 - 4/\kappa)A = 0.$$

This means (see Appendix B) in particular that A is a linear combination of the same function f as in Section 4, i.e.,

$$f(x) := F(4/\kappa, 1 - 4/\kappa, 8/\kappa; x)$$

and of another function that diverges like $x^{1-8/\kappa}$ as $x \rightarrow 0^+$. By the boundary conditions for Q , we conclude that A is a multiple of f and more precisely that $A(x) = f(x)/f(1)$, so that

$$Q(x) = x^{2/\kappa} f(x)/f(1).$$

Recall that our goal is to estimate $\mathbf{P}[C(1 - x)] = 1 - Q(x)$ as $x \rightarrow 1$. For this purpose, we express (via the connection formula (B.2)) the hypergeometric function f as a linear combination of the two natural independent hypergeometric functions that solve the same ODE in the neighborhood of 1, i.e., we write f as a linear combination of

$$f_1(x) := F(4/\kappa, 1 - 4/\kappa, 2 - 8/\kappa; 1 - x)$$

and of

$$f_2(x) := (1 - x)^{8/\kappa - 1} F(4/\kappa, 12/\kappa - 1, 8/\kappa; 1 - x)$$

and we get

$$f(x) = f(1)f_1(x) - \eta f_2(x),$$

where

$$\eta := -\frac{\Gamma(8/\kappa)\Gamma(1 - 8/\kappa)}{\Gamma(4/\kappa)\Gamma(1 - 4/\kappa)} = \frac{1}{-2\cos(4\pi/\kappa)}.$$

Hence, we see that as $x = 1 - \varepsilon \rightarrow 1$,

$$f(1 - \varepsilon) = f(1) - \varepsilon f'(1) - \eta \varepsilon^{8/\kappa - 1} + O(\varepsilon^2)$$

(recall that $1 \leq 8/\kappa - 1 < 2$ because $\kappa \in (8/3, 4]$). We can note that $f'(1) = -2f(1)/\kappa$ (which follows for example from (7.1)), so that

$$\frac{f(1 - \varepsilon)}{f(1)} = 1 + \frac{2}{\kappa} \varepsilon - \frac{\eta}{f(1)} \varepsilon^{8/\kappa - 1} + O(\varepsilon^2).$$

If we now expand $Q(1 - \varepsilon) = (1 - \varepsilon)^{2/\kappa} f(1 - \varepsilon)/f(1)$ when $\varepsilon \rightarrow 0$, we get that

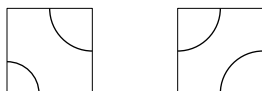
$$\mathbf{P}[C(\varepsilon)] = 1 - Q(1 - \varepsilon) \sim \frac{\eta}{f(1)} \varepsilon^{8/\kappa - 1},$$

which concludes the proof.

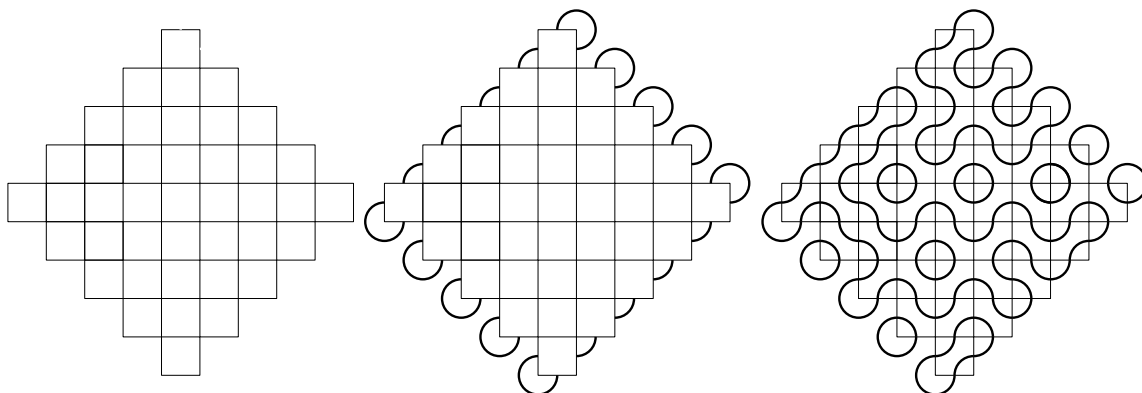
APPENDIX A. CONNECTION PROBABILITIES FOR DISCRETE $O(N)$ MODELS

Let us very quickly browse through the properties about hook-up probabilities in squares of discrete $FK(q)$ percolation models and of $O(N)$ models that we have been referring to in this paper. All these facts are elementary and classical (the reader can consult for instance [13] and the references therein).

We will first describe the example of the fully packed version of the $O(N)$ model on the square lattice (which is in fact directly related to the critical FK model on the square lattice for $q = N^2$). This fully-packed $O(N)$ model is the model where each small square in the domain is filled with one of the two possible options depicted in Figure 16.

FIGURE 16. The two tiles for the fully packed $O(N)$ model

Then, when one sets boundary conditions as in the middle of Figure 17, one gets a collection of loops as in the right of Figure 17. In the fully-packed $O(N)$ model, the probability of a configuration is chosen to be proportional to N^L where L is the number of loops in the configuration. When one

FIGURE 17. The fully-packed $O(N)$ model (with free boundary conditions)

explores the tiles of the $O(N)$ model starting from two corners, exploring the loops a Markovian way, one ends up with a configuration as in Figure 18. The conditional distribution of the remaining-to-be discovered configuration is now the discrete analog of our CLE with two wired boundary conditions. Examples of the fully-packed $O(N)$ model with two wired boundary parts in the original square correspond to the choice of boundary conditions depicted in the left of Figure 19, where the probability of a configuration is still proportional to the number of created loops (also taking into account the loops that go through the boundary).

We can note that when one is given a configuration for which the two boundary arcs are joined together into a single loop, then if one rotates the configuration by 90 degrees without rotating the boundary conditions, one has a configuration with exactly one more loop. It therefore follows that the probability that the two boundary arcs are joined into a single loop is N times smaller than the probability that they are part of two different loops. In other words, in Figure 19, the probability of the event that the two wired boundary arcs are part of the same loop is $1/(1 + N)$.

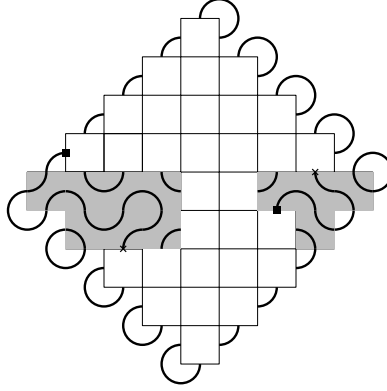


FIGURE 18. Exploring the fully packed $O(N)$ model leads to an $O(N)$ model with two (longer) wired arcs.

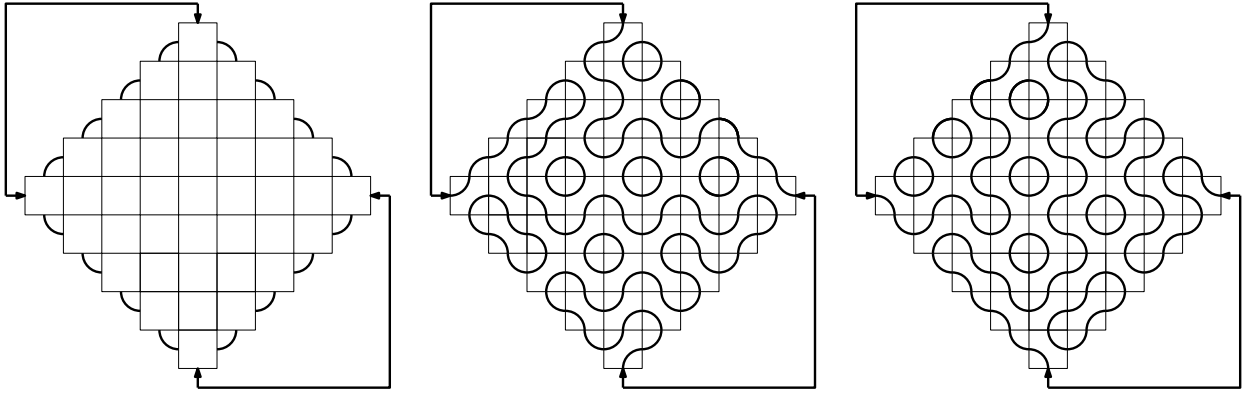


FIGURE 19. The boundary conditions (left). Rotating the middle configuration by 90 degrees creates exactly one additional loop. The probability that the boundary arcs hook up into one single loop (as in the middle picture) is $1/(1 + N)$.

A variation of the previous fully-packed loop model is to allow for additional configurations. This time, one considers the square as on the left-hand side of Figure 17, and an admissible configuration is when one fills each tile with one of the seven tiles depicted in Figure 20, in such a way that one only creates closed loops. One can then choose a parameter μ , and weight each configuration

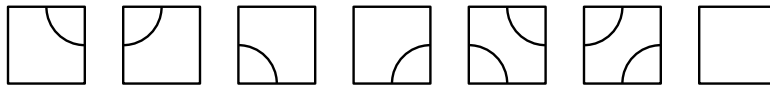


FIGURE 20. The seven tiles

by $N^L \mu^l$ where l denotes the cumulated length of all the loops. The previous fully-packed case corresponds to the limit when $\mu \rightarrow \infty$. Then exactly the same arguments lead to the definition of the corresponding model in the square with two wired boundaries, as depicted in Figure 21, and to the fact that for this model the probability that the two boundary arcs are part of the same loop is also $1/(1 + N)$, regardless of μ . Note that this property of $O(N)$ models is actually quite robust and works also for more general models as long as the probability of a configuration is the product of local weights times N^L . It also holds on other lattices, as long as they have enough symmetries.

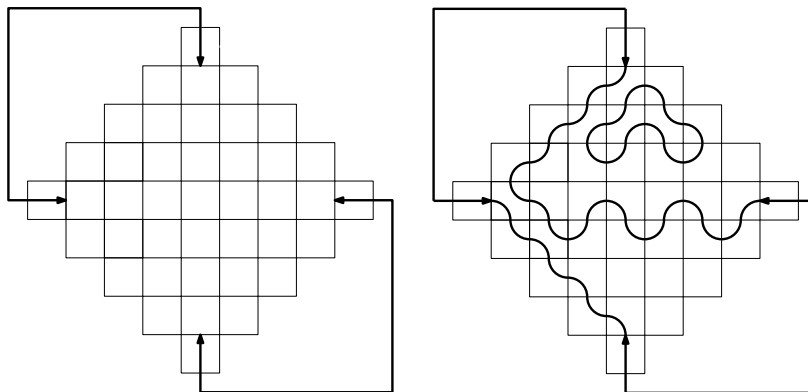


FIGURE 21. The boundary conditions (left). The probability that the boundary arcs hook up into one single loop is again $1/(1 + N)$.

The asymptotic behavior of the $O(N)$ model depends on the choice of μ . It is conjectured (see [20, 49]) that when $N \leq 2$, for a well-chosen critical value $\mu(N)$, it converges to a simple CLE_κ , while for large μ , it converges to a non-simple CLE_κ .

For the FK-percolation model on the square lattice (see for instance [19] and the references therein for its definition and basic properties), the corresponding crossing property can be stated as follows. Consider $q > 0$, and the $FK(q)$ model on the rectangle $[0, n+1] \times [0, n]$, where the left-hand boundary is wired and the right-hand boundary is wired (i.e., all points on the left boundary are identified as one single point, and all points on the right hand boundary are identified as another point). Consider also the self-dual value of the parameter p , i.e., take $p = \sqrt{q}/(1 + \sqrt{q})$. Then, the probability of a left-to-right crossing of this rectangle is $1/(1 + \sqrt{q})$.

One way to see it is via the usual duality trick because the dual configuration w^* to a configuration w is also a critical $FK(q)$ model on the dual graph, which is the rectangle $[1/2, N + 1/2] \times [-1/2, N + 1/2]$ but with the top and bottom sides identified as one single site (not two as for the left and right boundaries before). Hence, it follows exactly the same law as w rotated by 90 degrees, except that the configurations get an extra weight $1/q$ when there is no top to bottom crossing. Hence, if π is the probability of a left to right crossing for w , one has $1 - \pi = \pi/(\pi + (1 - \pi)/q)$ from which the statement follows. Another simple way is just to note that if we rotate the picture by 45 degrees, and look at the union of the outer boundaries of the collection of clusters and of the outer boundaries of dual clusters, one gets exactly the previous fully-packed $O(N)$ model with $N = \sqrt{q}$, with boundary conditions just as described in the $O(N)$ case above, so one can apply directly the previous considerations on fully-packed $O(N)$ models. See also for instance Section 2 of [53], or [13].

APPENDIX B. HYPERGEOMETRIC FUNCTIONS

For the convenience of those readers who are not so acquainted with the basic properties of hypergeometric functions that we are using (or to refresh their memories), we try to very briefly recall them in the following page. When a, b, c are real numbers, the hypergeometric function $F(a, b, c, ; z)$ is defined for all z in the open unit disk by the power series

$$F(a, b, c, ; z) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

where $(a)_n = a(a+1)\dots(a+n-1)$ is the rising Pochhammer symbol (with the convention $(a)_0 = 1$). When $c > a+b$ (which is in fact the case for all hypergeometric functions that we write out explicitly as a function of κ in this paper) this series converges also at $z = 1$ and the function is continuous on the interval $[0, 1]$. The value at 1 can then be expressed in terms of the Γ function:

$$F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

When $a+b > c$, then the hypergeometric function $F(a, b, c; x)$ diverges like a constant times $(1-x)^{c-a-b}$ when $x \rightarrow 1^-$. Indeed, one can check that

$$F(a, b, c; x) = (1-x)^{c-a-b} F(c-a, c-b, c; x).$$

The hypergeometric function $F(a, b, c; x)$ is a solution of the hypergeometric differential equation

$$(B.1) \quad x(1-x)f'' + (c-(a+b+1)x)f' - abf = 0$$

on the interval $(0, 1)$. Conversely, it is easy to check that when c is not a non-negative integer, any solution to this equation on the interval $(0, 1)$ is a linear combination of the two functions $F(a, b, c; x)$ and $x^{1-c}F(1+a-c, 1+b-c, 2-c; x)$.

If we use the change of variables $y = 1-x$, we can note that the equation (B.1) gets transformed into another hypergeometric equation. It therefore follows that a solution to (B.1) on $(0, 1)$ is also necessarily a linear combination of the two functions

$$\begin{aligned} f_1(x) &:= F(a, b, a+b+1-c; 1-x) \\ f_2(x) &:= (1-x)^{c-a-b} F(c-a, c-b, 1+c-a-b; 1-x). \end{aligned}$$

In particular, using the particular values of those functions at 0 and 1, one gets that on $(0, 1)$,

$$(B.2) \quad F(a, b, c; x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} f_1(x) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} f_2(x),$$

which is one of the connection formulas between hypergeometric functions (this is for instance 15.3.6 in [1]).

Similarly, if one looks for solutions to (B.1) on the interval $(1, \infty)$, one can use the change of variables $y = 1/x$ and see that when $a-b$ is not an integer, such a solution is necessarily a linear combination of the two functions

$$\begin{aligned} h_1(x) &:= x^{-a} F(a, 1+a-c, 1+a-b; 1/x), \\ h_2(x) &:= x^{-b} F(b, 1+b-c, 1+b-a; 1/x). \end{aligned}$$

Again, one can see that when $x \in (1, 2)$ such a solution is also a linear combination of f_1 and

$$\tilde{f}_2(x) := (x-1)^{c-a-b} F(c-a, c-b, 1+c-a-b; 1-x).$$

In particular,

$$(B.3) \quad h_1(x) = \frac{\Gamma(a-b+1)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)} f_1(x) + \frac{\Gamma(a-b+1)\Gamma(a+b-c)}{\Gamma(a)\Gamma(a-c+1)} \tilde{f}_2(x),$$

which is the other connection formula that we use in this paper (note that it describes in particular the precise asymptotic expansion of $h_1(x)$ in the limit when $x \rightarrow 1^+$).

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